



Advanced D-Partitioning Stability Analysis of Digital Control Systems with Multivariable Parameters

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Abstract

This paper contributes in further advancement of the method of D-Partitioning stability analysis applied to digital control system with multivariable parameters. It also explores the effects of simultaneous system uncertainties by determining graphically regions of stability in the space of the system's parameters. The Euler's approximation is taken into account and the Bilinear Tustin Transform method is employed as one of the most precise facilities for systems discretization. The interaction between the variable parameters will bring a new light in the graphical solution of the problem of control system stability, where the analysis of control systems with two and three simultaneously variable parameters is taken into consideration. The proposed research is a continuation and further development of the already published author's work on the advancement of the method of the D-Partitioning stability analysis of digital control systems with variable parameters. The suggested implementation for system's stability analysis is fundamental for the further advancement of control theory in this area.

Keywords: *Multivariable Parameters, Stability Analysis Tool, D-Partitioning, Interaction, Regions of Stability;*

Nomenclature

$G_O(s)$	Open-loop transfer function of the system
$G(s)$	Characteristic equation of the continuous system
s	Laplace operator
K	Variable gain of the control system
T_1, T_2, T_3	Variable time constants of the control system
T_s	Sampling period
T_{min}	Minimum time-constant of the system
ω_s	Sampling frequency
z	z-transform operator
μ, γ	System's variable parameters
2-D	Two-dimensional
3-D	Three-dimensional
GM	Gain margin
PM	Phase margin

1. Introduction

The D-partitioning method enables a quick and convenient determination of the regions of stability in case of variation of system's parameters. Following some initial ideas of Neimark [1], the method was better clarified and further developed by the author in previous published work [2], [3], [4], [5], [6], [7], now calcified as Advanced D-Partitioning. After its further expansion, the method became a powerful tool for system stability analysis, having a number of advantages compared to other well-known stability analysis methods. It has the benefit of a clear graphical display of the variation of each parameter and its effect on the system's stability.

The current research is extending the strategy for stability analysis of digital control systems with multivariable parameters, suggesting a technique for solving the problem by applying the Advanced D-Partitioning method in the discrete time-domain. The Advanced D-Partitioning analysis of digital control systems with multivariable parameters is achieved by introducing discretization based on the Bilinear Tustin Transform [8], while the systems' sampling time is chosen, considering the Euler's approximation [9], [10].

Exploring control systems with simultaneous variation of multiple parameters was inspired by cases of abnormal behaving systems, being stable at lower parameter values, further becoming unstable at a specific parameter range and turn out to be again stable again at higher parameter values [11], [12].

For the clarity of the presented research, this paper is organized in the following sequence. Initially, the nature of the D-Partitioning as stated by Neimark is explained. This is followed by further upgrading of the Advanced D-Partitioning analysis suggested by the author in his previous published work, now extended in the discrete time-domain. The results, achieved from the Advanced D-Partitioning analysis in the discrete-time domain for the cases of one, two and three variable parameters, are confirmed by the Nyquist and Bode stability criteria. The employed stability analysis methods are finally compared for the considered control systems variable parameters.



2. The Nature of the D-Partitioning as Originally Suggested by Neimark

The position of the characteristic equation roots in the s-plane depends on the values of the system's parameters. On this basis, Neimark suggested that the space of an n -order system's characteristic equation coefficients can be partitioned into a number of regions corresponding to the number of roots in the left-hand side of the s-plane. This method was categorized as D-Partitioning. At its initial steps the method was stating rather its theoretical possibilities and its applications were quite limited. Although, Neimark expanded his initial ideas, they were rarely implemented because of their obscurity.

3. Advanced D-Partitioning Stability Analysis (Case of One Variable Parameter)

As already indicated, in previous published work, the author of this research suggested further advancement of the method of the D-Partitioning stability analysis of continuous control systems by one variable parameter. To facilitate and realize the stability analysis, the system's characteristic equation is presented in the format [2], [3]:

$$G(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_n = 0 \quad (1)$$

Equation (1) can be also introduced in the following format to expose the variable parameter:

$$G(s) = P(s) + vQ(s) = 0 \quad (2)$$

The D-partitioning regions could be obtained by substituting $s = j\omega$.

$$G(j\omega) = P(j\omega) + vQ(j\omega) = 0 \quad (3)$$

Therefore, the variable parameter can be presented as a complex number:

$$v = -\frac{P(j\omega)}{Q(j\omega)} = X(\omega) + jY(\omega) \quad (4)$$

The D-partitioning regions are obtained graphically in the v-plane by allocating values of the frequency within the range $-\infty \leq \omega \leq +\infty$.

Since, in the s-plane, the region of stability remains on the left-hand side of the plane, in the complex plane $v = X(\omega) + jY(\omega)$, a region of stability remains always on the left-hand side of the D-Partitioning curve for a change of frequency from $-\infty$ to $+\infty$.

The D-partitioning curve in terms of one variable parameter can be plotted in the complex plane within the frequency range $-\infty \leq \omega \leq +\infty$, facilitated by MATLAB the "nyquist" m-code. The procedure can work on any computer where the MATLAB program is installed.

To avoid any misunderstanding or misinterpretation of the D-Partitioning procedure, the "nyquist" m-code is modified into a "dpartition" m-code with the aid of the MATLAB Editor and a proper formatting. The "dpartition" m-code will plot the curve of a specific system parameter in terms of the frequency variation from $-\infty$ to $+\infty$.

The results obtained by the "dpartition" m-code are the same as those achieved by the "nyquist" m-code. This D-Partitioning procedure can work only on a computer where the new developed "dpartition" m-code is included in the MATLAB program.

In order to benefit from the Advanced D-Partitioning analysis shown in this research, the wider engineering community can still use the "nyquist" m-code for the purpose of plotting the D-partitioning curve.

The Advanced D-Partitioning analysis in case of one variable parameter is illustrated in this research by a system consisting of an armature-controlled dc motor and a type-driving mechanism is suggested [11], [12], [13]. Initially, it is considered that the system is experiencing variation of only one of its parameters. Usually, the variation of the load causes variation of the mechanism time-constant T_l . The transfer function of the open loop system in the continuous-time domain is presented as:

$$G_{ol}(s) = \frac{10}{(1 + T_1 s)(1 + 0.5s)(1 + 0.8s)} = \frac{10}{T_1 (0.4s^3 + 1.3s^2 + s) + 0.4s^2 + 1.3s + 1} \quad (5)$$

The characteristic equation of the continuous stand-alone unity feedback system is:

$$G(s) = T_1 (0.4s^3 + 1.3s^2 + s) + 0.4s^2 + 1.3s + 11 = 0 \quad (6)$$

The regions of stability reflecting the variation of the system's parameter T_l are determined by implementing the method of the Advanced D-Partitioning. The time-constant T_l is presented from equation (6) as follows:

$$T_1(s) = -\frac{T_1 T_2 s^2 + (T_1 + T_2)s + K + 1}{T_1 T_2 s^3 + (T_1 + T_2)s^2 + s} = -\frac{0.4s^2 + 1.3s + 11}{0.4s^3 + 1.3s^2 + s} \quad (6)$$

The variable $T_l(s)$ is initially introduced as a continuous-time function and next converted into its digital equivalent $T_l(z)$, facilitated by the Tustin Transform [14], [15], [16]. In accordance with the Euler's approximation, observing that $T_s \leq (0.1T_{min} \text{ to } 0.2T_{min})$, the sampling period is chosen as $T_s = 0.05\text{sec}$, since the continuous system minimum time-constant is $T_{min} = 0.5\text{sec}$. Then, the D-Partitioning in terms of the variable time-constant $T_l(z)$ is achieved in the discrete-time domain by the code:

```
>> T1 = tf([-0.4 -1.3 -11],[0.4 1.3 1 0])
Transfer function:
-0.4 s^2 - 1.3 s - 11
-----
0.4 s^3 + 1.3 s^2 + s
>> T1d = c2d(T1,0.05,'tustin')
Transfer function:
-0.02536 z^3 + 0.02002 z^2 + 0.02377 z - 0.02161
-----
z^3 - 2.844 z^2 + 2.694 z - 0.8499
Sampling time: 0.05
>> dpartition(T1d)
```



From where:

$$T_1(z) = -\frac{-0.02536z^3 + 0.02002z^2 + 0.02377z - 0.02161}{z^3 - 2.844z^2 + 2.694z - 0.8499} \quad (7)$$

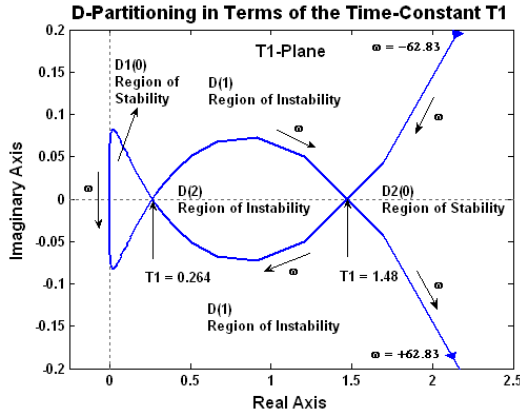


Figure 1. Advanced D-Partitioning Analysis of the Digital Control System in Terms of the Variable Time-Constant T_1

As seen from Figure 1, the Advanced D-Partitioning determines four regions in the complex T_1 -Plane: D1(0), D2(0), D(1) and D(2). Only D1(0) and D2(0) are regions of stability, since the position of these regions is always on the left-hand side of the D-Partitioning curve within the range of $\omega = \pm \omega_s/2 = \pm 2\pi/2T_s = \pm 62.83$ rad/sec [17]. Only the real values of T_1 are considered, hence the regions of stability are reduced to lines of stability.

4. System Evaluation with Nyquist and Bode Stability Criteria (Case of One Variable)

If the time-constant is within $0 \text{ sec} \leq T_1 \leq 0.264 \text{ sec}$, corresponding to the region of stability D1(0), the system will be stable. If T_1 is varied within $0.264 \text{ sec} \leq T_1 \leq 1.48 \text{ sec}$, the system will be unstable, being within the region of instability D(2). If $T_1 > 1.48 \text{ sec}$, the system is again stable and is operating in the region of stability D2(0). If the time-constant is $T_1 = 0.264 \text{ sec}$ or $T_1 = 1.48 \text{ sec}$, the digital closed-loop control system becomes marginal.

The open-loop transfer functions for both marginal cases of the control system in the continuous time-domain are determined accordingly:

$$G_{O.0.264}(s) = G_{O1}(s) = \frac{10}{(1 + 0.264s)(1 + 0.5s)(1 + 0.8s)} = \frac{10}{0.1056s^3 + 0.7432s^2 + 1.564s + 1} \quad (8)$$

$$G_{O.1.48}(s) = G_{O1}(s) = \frac{10}{(1 + 1.48s)(1 + 0.5s)(1 + 0.8s)} = \frac{10}{0.592s^3 + 2.324s^2 + 2.78s + 1} \quad (9)$$

The Advanced D-Partitioning in the discrete-time domain is confirmed with the Nyquist stability criterion and the Bode stability criterion for the two marginal values of T_1 . They are introduced into the system's continuous transfer functions that are further converted into their digital equivalents [8], [13], [14], [17] with the Bilinear Tustin Transform. This is facilitated by the following code:

```
>> Go1=tf([0 10],[0.1056 0.7432 1.564 1])
Transfer function:
10
-----
0.1056 s^3 + 0.7432 s^2 + 1.564 s + 1
>> God1=c2d(Go1,0.05,'tustin')
Transfer function:
0.001248 z^3 + 0.003745 z^2 + 0.003745 z + 0.001248
-----
z^3 - 2.671 z^2 + 2.375 z - 0.7029
Sampling time: 0.05
>> Go2=tf([0 10],[0.592 2.324 2.78 1])
Transfer function:
10
-----
0.592 s^3 + 2.324 s^2 + 2.78 s + 1
>> God2=c2d(Go2,0.05,'tustin')
Transfer function:
0.0002397 z^3 + 0.0007191 z^2 + 0.0007191 z + 0.0002397
-----
z^3 - 2.811 z^2 + 2.633 z - 0.8217
Sampling time: 0.05
>> nyquist(God1,God2)
>> bode(God1,God2)
>> margin(Gpod1)
GM = 0.00652 dB, PM = 0.0193 deg
>> margin(Gpod2)
GM = - 0.0756 dB, PM = - 0.276 deg
```

As seen, after applying the Bilinear Tustin Transform, the digital equivalents of equations (8) and (9) are reflected in equations (10) and (11), as seen from the code above.

$$G_{Od.0.264}(z) = G_{od1}(z) = \frac{0.001248z^3 + 0.003745z^2 + 0.003745z + 0.001248}{z^3 - 2.671z^2 + 2.375z - 0.7029} \quad (10)$$

$$G_{Od.1.48}(z) = G_{od2}(z) = \frac{0.0002397z^3 + 0.0007191z^2 + 0.0007191z + 0.0002397}{z^3 - 2.811z^2 + 2.633z - 0.8217} \quad (11)$$

As a result of the applied code, Figure 2 demonstrates the Nyquist stability criterion in the discrete-time domain for both of the cases related to marginal time-constant values $T_1 = 0.264 \text{ sec}$ and $T_1 = 1.48 \text{ sec}$.

Marginal Nyquist Diagrams in the Discrete Time-Domain
Cases of $T_1 = 0.264 \text{ sec}$ and $T_1 = 1.48 \text{ sec}$

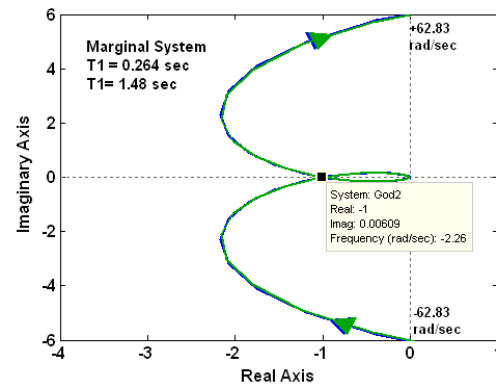


Figure 2. Confirmation of the Result of the Advanced D-Partitioning Analysis with the Aid of the Nyquist Stability Criterion in the Discrete-Time Domain (Marginal Cases $T_1 = 0.264 \text{ sec}$; $T_1 = 1.48 \text{ sec}$)



As seen from Figure 2, the Nyquist curves of the digital open loop system coincide and are passing approximately via the point $(-1, j0)$, hence the closed loop digital control system is marginal. This confirms the results, obtained from the Advanced D-Partitioning stability analysis.

The achieved results from the code, confirm also the outcome of the Advanced D-Partitioning analysis with the aid of the Bode stability criterion in the discrete-time domain for both of the cases related to marginal time-constant values $T_l = 0.264\text{sec}$ and $T_l = 1.48\text{sec}$.

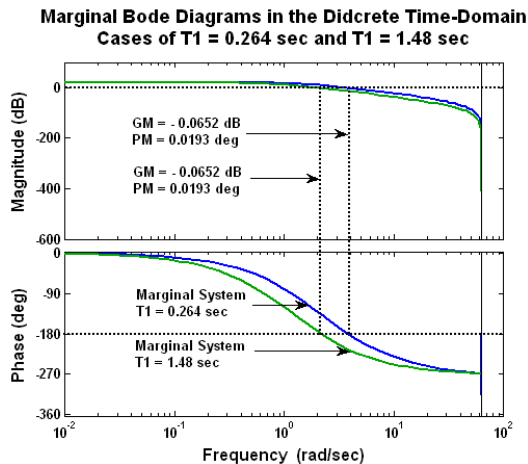


Figure 3. Confirmation of the Result of the Advanced D-Partitioning Analysis with the Aid of the **Bode Stability Criterion** in the Discrete-Time Domain (Marginal Cases $T_l = 0.264\text{sec}$; $T_l = 1.48\text{sec}$)

As shown in Figure 3, if the marginal value of the time-constant is $T_l = 0.264\text{ sec}$, the results for the digital system gain margin is $GM = 0.00652\text{ dB} \approx 0\text{ dB}$ and the system phase margin is $PM = 0.0193^\circ \approx 0^\circ$. The results also prove the second marginal value of the time-constant $T_l = 1.48\text{sec}$, where the gain margin is $GM = -0.0756\text{ dB} \approx 0\text{ dB}$ and phase margin is $PM = -0.276^\circ \approx 0^\circ$.

Again, the analysis of both Nyquist stability criterion and Bode stability criterion is restricted to the frequency $\omega = \pm \omega_s/2 = \pm 2\pi/2T_s = \pm 62.83\text{ rad/sec}$ [11], [12], [17].

The system stability within the regions D1(0) and D2(0) is explored in the discrete-time domain by allocating values for the variable time-constant $T_l = 0.1\text{ sec}$ and $T_l = 5\text{ sec}$ and applying the following code:

```
>> Go3=tf([0 10],[0.04 0.53 1.4 1])
>> God3=c2d(Go3,0.05,'tustin')
Transfer function:
0.002886 z^3 + 0.008658 z^2 + 0.008658 z + 0.002886
-----
z^3 - 2.444 z^2 + 1.956 z - 0.51
Sampling time: 0.05
>> Go4=tf([0 10],[2 6.9 6.3 1])
>> God4=c2d(Go4,0.05,'tustin')
Transfer function:
7.179e-005 z^3 + 0.0002154 z^2 + 0.0002154 z + 7.179e-005
-----
z^3 - 2.834 z^2 + 2.676 z - 0.8415
Sampling time: 0.05
>> nyquist(God3,God4)
>> bode(God3,God4)
>> margin(God3)
GM = 4.89 dB, PM = 15.5 deg
>> margin(God4)
GM = 6.34 dB, PM = 24 deg
```

Nyquist Diagrams in the Discrete Time-Domain
Cases of Stable System; $T_l = 0.1\text{ sec}$ and $T_l = 5\text{ sec}$

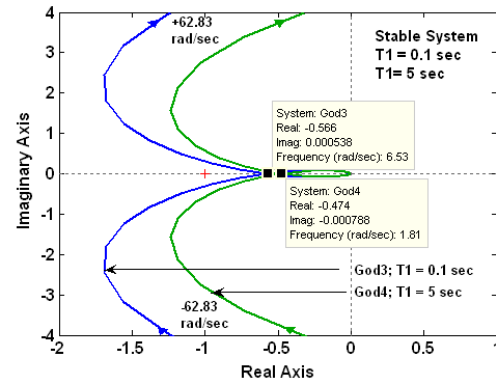


Figure 4. Confirmation of the Result of the Advanced D-Partitioning Analysis with the Aid of the **Nyquist Stability Criterion** in the Discrete-Time Domain (Stable Case $T_l = 0.1\text{sec}$; $T_l = 5\text{sec}$)

Since, the Nyquist curves of the digital open loop control system are passing approximately via the points at the real axes $(-0.566, j0)$ and $(-0.474, j0)$, the closed loop digital control system is stable for both cases, confirming the result from the Advanced D-Partitioning analysis.

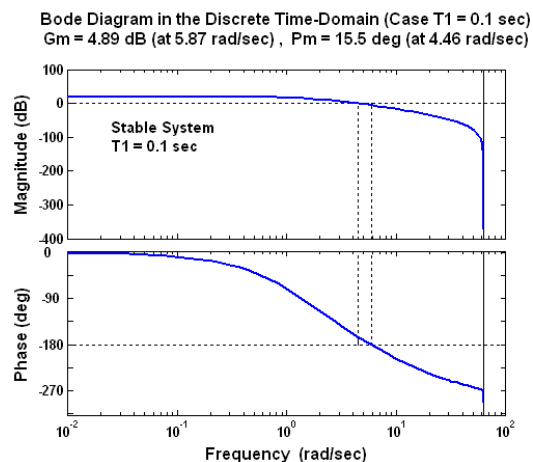


Figure 5. Confirmation of the Result of the Advanced D-Partitioning Analysis with the Aid of the **Bode Stability Criterion** in the Discrete-Time domain (Case of Stable System $T_l = 0.1\text{sec}$)

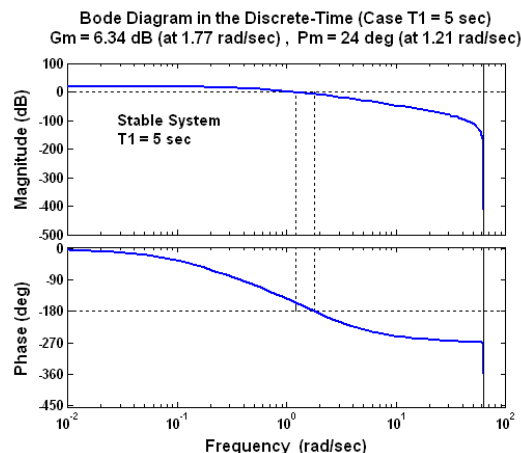


Figure 6. Confirmation of the Result of the Advanced D-Partitioning Analysis with the Aid of the **Bode Stability Criterion** in the Discrete-Time domain (Case of Stable System $T_l = 5\text{sec}$)



As seen from Figure 5 and Figure 6, and from the results of the code, the Advanced D-Partitioning analysis is confirmed with the aid of the Bode stability criterion in the discrete-time domain for both of the cases related to a stable system ($T_I = 0.1\text{sec}$ and $T_I = 5\text{sec}$). The positive gain and phase margins ($GM = 4.89\text{ dB}$ and $PM = 15.5^\circ$) and ($GM = 6.34\text{ dB}$ and $PM = 24^\circ$), prove that the digital system is stable within the regions D1(0) and D2(0).

Further, the system stability within the region D(2) is explored in the discrete-time domain by allocating a value for the variable $T_I = 1\text{ sec}$ and applying the code:

```
>> Go5=tf([0 10],[0.4 1.7 2.3 1])
>> God5=c2d(Go5,0.05,'tustin')
Transfer function:
0.000352 z^3 + 0.001056 z^2 + 0.001056 z + 0.000352
-----
z^3 - 2.795 z^2 + 2.604 z - 0.8085
Sampling time: 0.05
>> nyquist(God5)
>> bode(God5)
>> margin(God5)
GM = -1.14 dB, PM = -4.11deg
```

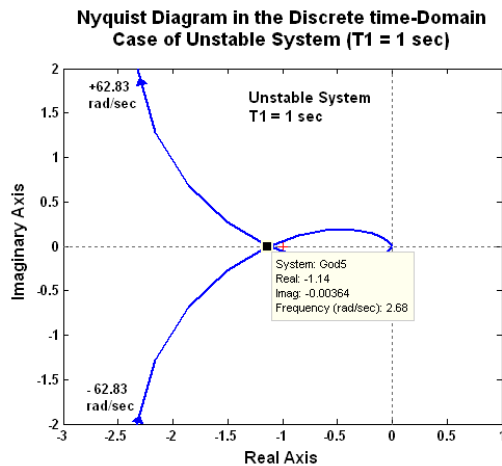


Figure 7. Confirmation of the Result of the Advanced D-Partitioning Analysis with the Aid of the **Nyquist Stability Criterion** in the Discrete-Time domain (Unstable Case $T_I = 1\text{ sec}$)

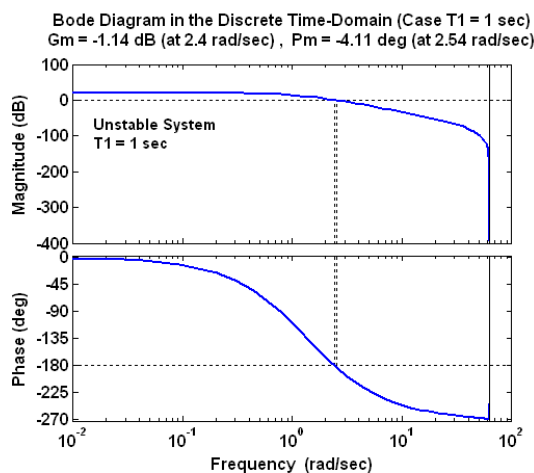


Figure 8. Confirmation of the Result of the Advanced D-Partitioning Analysis with the Aid of the **Bode Stability Criterion** in the Discrete-Time domain (Unstable Case $T_I = 1\text{ sec}$)

If applying the Nyquist stability criterion for the case $T_I = 1$, the Nyquist curve of the digital open loop control system is passing approximately via the points at the real axes $(-1.14, j0)$. Therefore, the closed loop digital control system is unstable.

By applying the Bode stability criterion for the case $T_I = 1$, the negative values of the digital system margins ($GM = -1.14\text{ dB}$ and $PM = -4.11^\circ$), confirm that the digital system is unstable within the region D(2).

In case of control systems with one variable parameter, the Advanced D-Partitioning method directly exposes the transparent images of regions of stability and instability, as well as the system's margins of stability in the complex plane of the variable parameter. It is also demonstrated that the Advanced D-Partitioning can be successfully applied to digital control systems. Both Nyquist and Bode stability criteria confirm the results from the Advanced D-Partitioning analysis.

5. Advanced D-Partitioning Stability Analysis (Case of Two Variable Parameters)

Further upgrade of the Advanced D-Partitioning analysis in case of systems with two simultaneously variable parameters [5], [6], [7] is suggested in this research. The system's characteristic equation generally is presented as:

$$G(s) = \mu P(s) + \gamma Q(s) + R(s) = 0 \quad (12)$$

where $P(s)$, $Q(s)$, and $R(s)$ are polynomials of s
 μ and γ are system's variables parameters

The border of the D-Partitioning regions in the plain (μ, γ) is determined by:

$$G(j\omega) = \mu P(j\omega) + \gamma Q(j\omega) + R(j\omega) = 0 \quad (13)$$

If a hypothetical third order unity feedback system of Type 0 is considered, the system's characteristic equation can be presented as:

$$G(s) = (T_1 s + 1)(T_2 s + 1)(T_3 s + 1) + K = 0 \quad (14)$$

It is suggested that two of the system's parameters are simultaneously variable:

$$T_I = T = \mu, \quad K = \gamma. \quad (15)$$

The objective is to determine the regions of variation of these two parameters, for which the system will be stable. Equations (15) are substituted in (14), from where:

$$\left. \begin{aligned} \mu [T_2 T_3 s^3 + (T_2 + T_3) s^2 + s] + \\ + \gamma + T_2 T_3 s^2 + (T_2 + T_3) s + 1 = 0 \end{aligned} \right\} \quad (16)$$

By substituting $s = j\omega$ in equation (16) and taking into account equation (13), equation (16) could be presented in the detailed form as set of equations (17):

$$\left. \begin{aligned} P(j\omega) &= [T_2 T_3 (j\omega)^3 + (T_2 + T_3) (j\omega)^2 + j\omega] \\ Q(j\omega) &= 1 \\ R(j\omega) &= T_2 T_3 (j\omega)^2 + (T_2 + T_3) j\omega + 1 \end{aligned} \right\} \quad (17)$$

Taking into consideration equations (13) and (17), the variable system's parameters can be determined as:



$$\left. \begin{aligned} \mu &= \frac{\Delta_1}{\Delta} = \frac{T_2 + T_3}{T_2 T_3 \omega^2 - 1} \\ \gamma &= \frac{\Delta_2}{\Delta} = \frac{(T_2 T_3 \omega^2 - 1)^2 + (T_2 + T_3)^2 \omega^2 + 1}{T_2 T_3 \omega^2 - 1} \end{aligned} \right\} \quad (18)$$

$$\text{or } \Delta = T_2 T_3 \omega^2 - 1 \quad (19)$$

The determinant Δ becomes $\Delta = 0$ at a specific frequency $\omega = \omega_\infty$ that can be found out from equation (19) as:

$$\omega = \omega_\infty = \sqrt{\frac{1}{T_2 T_3}} \quad (20)$$

If $\Delta = 0$, it is clear from equations (20) that at $\omega = \omega_\infty$ both system parameters are approaching infinity:

$$\mu(\omega_\infty) \rightarrow \infty, \quad \gamma(\omega_\infty) \rightarrow \infty, \quad (21)$$

This implies that the main D-Partitioning curve has an interruption, or a breakdown, at a frequency $\omega = \omega_\infty$. It consists of two parts, plotted within the frequency ranges $0 < \omega < \omega_\infty$ and $\omega_\infty < \omega < \infty$.

For a better clarification and simplicity, first the functions $\mu(\omega)$ and $\gamma(\omega)$ are plotted, as shown in Figure 9(a) and Figure 9(b).

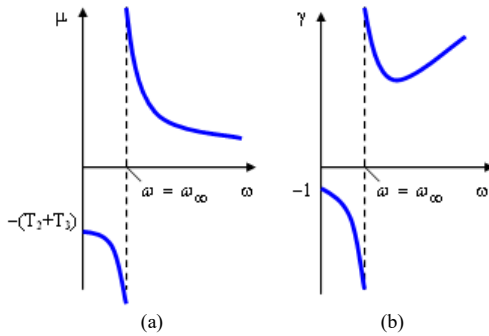


Figure 9. The Graphical Presentations of $\mu(\omega)$ and $\gamma(\omega)$ Showing the Interruption of the Curves at a Frequency $\omega = \omega_\infty$

The regions of the D-Partitioning also depend on straight lines in the (μ, γ) plane, known as *special lines*. The special lines are plotted for two border frequencies $\omega = 0$ and $\omega = +\infty$. Then the coefficients a_n and a_o of the equation (1) depend directly on the parameters μ and γ and the equations of the special lines are obtained by:

$$\left. \begin{aligned} a_n &= \mu T_2 T_3 = 0, & a_o &= \gamma + 1 = 0 \\ \mu &= 0 = \text{Const} & \gamma &= -1 = \text{Const} \end{aligned} \right\} \quad (22)$$

By combining the curves $\mu(\omega)$, $\gamma(\omega)$ and the special lines, the D-Partitioning is obtained in the (μ, γ) plane as seen in Figure 10. Taking into account that $\mu = T_1 = T$ is a time-constant and it can adopt only positive values, only the stable region $D_1(0)$ should be considered. The region of stability $D_1(0)$ is locked within the left-hand side of the D-Partitioning curve, corresponding to frequency rise from $\omega = \omega_\infty$ to $\omega \rightarrow \infty$ and the special line $\gamma = -1$. Since the gain $K = \gamma$ may also adopt only positive values, the realistic border of the stable region $D_1(0)$ should be considered $K = \gamma = 0$.

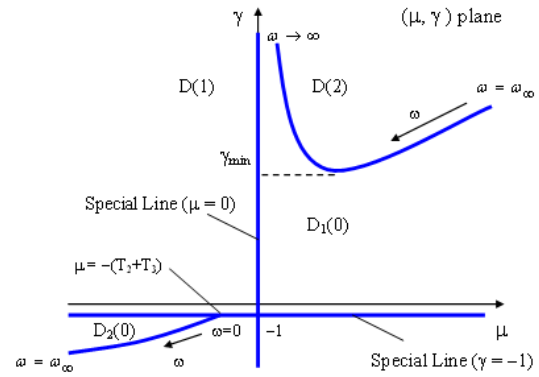


Figure 10. The Regions of Stability $D_1(0)$ and $D_2(0)$ Locked between the D-Partitioning Curves and the Special Lines

If the gain $K < \gamma_{min}$, the system is stable for any values of the time-constant $\mu = T_1$. For $K > \gamma_{min}$, the system is stable only for very small or very large values of the time-constant $\mu = T_1$. This case is demonstrated for the system of the armature-controlled dc motor and a type-driving mechanism with open-loop transfer function:

$$G_{O2}(s) = \frac{K}{(1 + Ts)(1 + 0.5s)(1 + 0.8s)} \quad (23)$$

The characteristic equation of the feedback system is:

$$K + (1 + Ts)(1 + 0.5s)(1 + 0.8s) = 0 \quad (24)$$

By substituting $s = j\omega$ equation (24) is modified to:

$$K = -1 + (1.3T + 0.4)\omega^2 + j\omega(0.4T\omega^2 - 1.3 - T) \quad (25)$$

Since the gain may obtain only real values, the imaginary term of equation (25) is set to zero, from where:

$$\omega^2 = \frac{1.3 + T}{0.4T} \quad (26)$$

The result of (26) is substituted into the real part of equation (25), from where:

$$K = \frac{1.3T^2 + 1.69T + 0.52}{0.4T} = 3.25T + 4.225 + \frac{1.3}{T} \quad (27)$$

When considering the continuous control system with the transfer function of equation (23), the D-Partitioning curve $K = f(T)$ is plotted with the aid of the code:

```
>> T=0:0.1:5;
>> K=3.25.*T+4.225+1.3./T
K=
Columns 1 through 10
Inf 17.5500 11.3750 9.5333 8.7750 8.4500 8.3417
8.3571 8.4500 8.5944
Columns 11 through 20
8.7750 8.9818 9.2083 9.4500 9.7036 9.9667 10.2375
10.5147 10.7972 11.0842
Columns 21 through 30
11.3750 11.6690 11.9659 12.2652 12.5667 12.8700
13.1750 13.4815 13.7893 14.0983
Columns 31 through 40
14.4083 14.7194 15.0313 15.3439 15.6574 15.9714
16.2861 16.6014 16.9171 17.2333
Columns 41 through 50
17.5500 17.8671 18.1845 18.5023 18.8205 19.1389
19.4576 19.7766 20.0958 20.4153
Column 51
20.7350
>> plot(T,K)
```



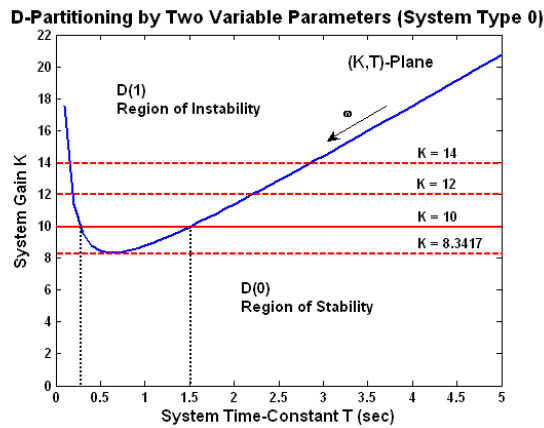


Figure 11. D-Partitioning in Terms of Two Variable Parameters

As seen from Figure 11, if $K < 8.3417$, a limit determined with the aid of MATLAB interactive procedure, the system is stable for any value of the T . The system performance and stability depends on the interaction between the two simultaneously varying parameters.

Further, a very important phenomenon is demonstrated in Figure 11. It is seen that the D-Partitioning curve $K = f(T)$ defines the border between the region of stability D(0) and instability D(1) for the case of simultaneous variation of the two system parameters. **Each point of the D-Partitioning curve represents the simultaneous marginal values of the two variable parameters. This result is a unique advancement and innovation in the theory of control systems stability analysis.**

The system performance is investigated for any values of the variable gain K , like $K = 10$, $K = 12$, $K = 14$. If K is varied, this affects the values of T at which the system may become unstable. Higher values of the gain K , enlarges the range of the time-constant T at which the system will fall into instability.

The parameters interaction of the continuous control system is demonstrated in case of variation of the time-constant T , when the gain is set to $K = 10$. If the time-constant range is $0 < T < 0.25$ sec, or $T > 1.5$ sec the system is stable. The system becomes unstable in the time-constant range $0.25 \text{ sec} < T < 1.5$ sec.

If the results of the continuous control system are compared with the results for the same system in the digital time-domain, as demonstrated in Figure 1, it is obvious that there is a very close match. In the digital time-domain within the range of $\omega = \pm \omega_s/2 = \pm 2\pi/2T_s$, if the gain is set to $K = 10$, the system is stable for the time intervals $0 < T < 0.264$ sec, or $T > 1.48$ sec. It is unstable in the range $0.264 \text{ sec} < T < 1.48$ sec. This insignificant difference of the results is due to the implementation of the Euler's approximation.

6. System Evaluation with Nyquist and Bode Stability Criteria (Case of Two Variables)

The performance of the control system with two variable parameters can be tested with the Nyquist stability criterion and the Bode stability criterion in the discrete-time domain for any of the simultaneous marginal values

of the variables T and K . Initially, the system's transfer function is introduced as a continuous-time function and next it is converted into its digital equivalent.

By substituting any two simultaneous marginal values of the two variable parameters, for instance, time-constant $T = 0.25$ sec and gain $K = 10$ in equation (23) results in:

$$G_o(s) = \frac{10}{(1+0.25s)(1+0.5s)(1+0.8s)} = \frac{10}{0.1s^3 + 0.725s^2 + 1.55s + 1} \quad (28)$$

Again, taking into account the Euler's approximation, $T_s \leq (0.1T_{min} \text{ to } 0.2T_{min})$, where the system's minimum time-constant is $T_{min} = 0.5$ sec and the sampling period is chosen as $T_s = 0.05$ sec. The system's stability assessment in the discrete-time domain is achieved by the code:

```
>> Go21=tf([0 10],[0.1 0.725 1.55 1])
Transfer function:
      10
-----
0.1 s^3 + 0.725 s^2 + 1.55 s + 1
>> God21=c2d(Go21,0.05,'tustin')
Transfer function:
0.001312 z^3 + 0.003935 z^2 + 0.003935 z + 0.001312
-----
z^3 - 2.662 z^2 + 2.359 z - 0.6954
Sampling time: 0.05
>> nyquist(God21)
>> margin(God21)
GM = 0.017 dB, PM = 0.028 deg
```

When applying the Nyquist stability criterion, as seen from Figure 12, the Nyquist curve of the digital open loop system is passing approximately via the point $(-1, j0)$, therefore the closed loop digital control system is marginal. This confirms the results, obtained from the Advanced D-Partitioning stability analysis for the marginal system with two variable parameters.

Prove of the D-Partitioning Analysis for the Case of Two Simultaneously Marginal Parameter Values $T = 0.25$ sec; $K = 10$

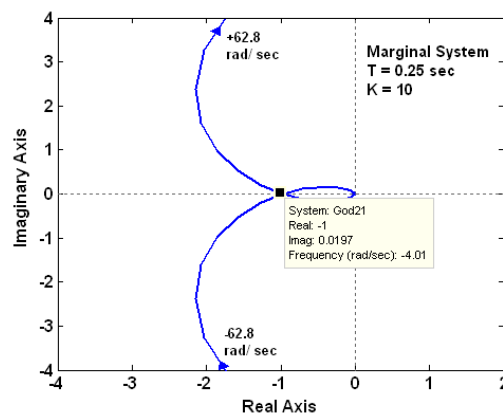


Figure 12. Confirmation of the result of the Advanced D-Partitioning Analysis with the Aid of the Nyquist Stability Criterion in the Discrete-Time domain (Marginal Case $T = 0.25$ sec; $K = 10$)

As seen from the Bode diagram presented at Figure 13, the system is explored in the discrete-time domain for $T = 0.25$ sec and $K = 10$. The achieved results for the gain and phase margins are $GM = 0.204 \text{ dB} \approx 0 \text{ dB}$ and $PM = 0.729^\circ \approx 0^\circ$, proving that the system is marginal.



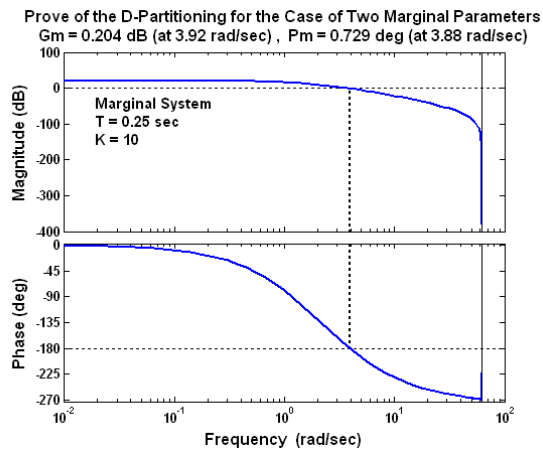


Figure 13. Prove of the D-Partitioning Analysis with the aid of the Bode Stability Criterion – Case of Two Simultaneously Marginal Values (Case $T = 0.25\text{sec}$ and $K = 10$)

The system stability within the region $D(0)$ is explored in the discrete-time domain if $T = 1$ sec and $K = 5$. The positive values for the gain and phase margins ($GM = 4.89$ dB and $PM = 20.6^\circ$), prove that the digital system is stable within the region $D(0)$, as seen from the code:

```
>> Go22=tf([0 5],[0.4 1.7 2.3 1])
Transfer function:
      5
-----
0.4 s^3 + 1.7 s^2 + 2.3 s + 1
>> God22=c2d(Go22,0.05,'tustin')
Transfer function:
0.000176 z^3 + 0.0005279 z^2 + 0.0005279 z + 0.000176
-----
z^3 - 2.795 z^2 + 2.604 z - 0.8085
Sampling time: 0.05
>> margin(God22)
GM = 4.89 dB, PM = 20.6 deg
```

The system instability within the region $D(1)$ is explored in the discrete-time if $T = 1$ sec and $K = 14$. The negative values of the digital system margins ($GM = -4.06$ dB and $PM = -13.7^\circ$), prove that the digital system is unstable within the region $D(1)$, as seen from the code:

```
>> Go23=tf([0 14],[0.4 1.7 2.3 1])
Transfer function:
      14
-----
0.4 s^3 + 1.7 s^2 + 2.3 s + 1
>> God23=c2d(Go23,0.05,'tustin')
Transfer function:
0.0004927 z^3 + 0.001478 z^2 + 0.001478 z + 0.0004927
-----
z^3 - 2.795 z^2 + 2.604 z - 0.8085
Sampling time: 0.05
>> margin(God23)
GM = -4.06 dB, PM = -13.7 deg
```

7. Advanced D-Partitioning Stability Analysis (Case of Three Variable Parameters)

Further advancement of the D-Partitioning is suggested in case of systems with three simultaneously variable parameters, where their interaction is presented in the 3-Dimensional (3-D) space. This is demonstrated for the control system of the armature-controlled dc motor and a type-driving mechanism, where the system's gain K and

two of the time-constants T_1 and T_2 are variable. The open-loop transfer function of the system is presented as:

$$G(s) = \frac{K}{(1 + T_1 s)(1 + T_2 s)(1 + 0.8s)} \quad (29)$$

The characteristic equation of the feedback system is:

$$K + (1 + T_1 s)(1 + T_2 s)(1 + 0.8s) = 0 \quad (30)$$

By substituting $s = j\omega$ equation (30) is modified to:

$$K = -1 + 0.8(T_1 + T_2)\omega^2 + j\omega(-T_1 - T_2 - 0.8 + 0.8T_1T_2\omega^2) \quad (31)$$

The imaginary term of equation (31) is set to zero, since the gain may obtain only real values, from where:

$$\omega^2 = \frac{T_1 + T_2 + 0.8}{0.8T_1T_2} \quad (32)$$

When the result of (32) is substituted into the real part of equation (31), results in:

$$K = -1 + 0.8(T_1 + T_2) + \frac{T_1 + T_2 + 0.8}{0.8T_1T_2} \quad (34)$$

The D-Partitioning plane is created in the (K, T_1, T_2) -Space of the variable parameters. The code is only partly presented, due to its considerable length:

```
>> [X,Y] = meshgrid(0:.15:3))
X =
Columns 1 through 10
0 0.1500 0.3000 0.4500 0.6000 0.7500 0.9000 1.0500
1.2000 1.3500
0 0.1500 0.3000 0.4500 0.6000 0.7500 0.9000 1.0500
1.2000 1.3500
.....
Columns 11 through 20
1.5000 1.6500 1.8000 1.9500 2.1000 2.2500 2.4000
2.5500 2.7000 2.8500
1.5000 1.6500 1.8000 1.9500 2.1000 2.2500 2.4000
2.5500 2.7000 2.8500
Y =
Columns 1 through 10
0 0 0 0 0 0 0 0 0 0
0.1500 0.1500 0.1500 0.1500 0.1500 0.1500 0.1500 0.1500
0.1500 0.1500 0.1500
0.3000 0.3000 0.3000 0.3000 0.3000 0.3000 0.3000 0.3000
0.3000 0.3000 0.3000
.....
Column 21
0
0.1500
0.3000
>> Z = -1+0.8.*(X+Y)+(X+Y+0.8)./(0.8.*X.*Y)
Warning: Divide by zero.
Z =
Columns 1 through 10
Inf Inf Inf Inf Inf Inf Inf Inf Inf
Inf 60.3511 34.0822 25.4059 21.1278 18.6089 16.9696
15.8330 15.0106 14.3975
Inf 34.0822 18.9244 13.9519 11.5256 10.1178 9.2193
8.6117 8.1861 7.8817
3.6249 3.5605

Column 21
Inf
12.4922
7.3344
.....
4.7444
>> surf(X,Y,Z,gradient(Z))
```



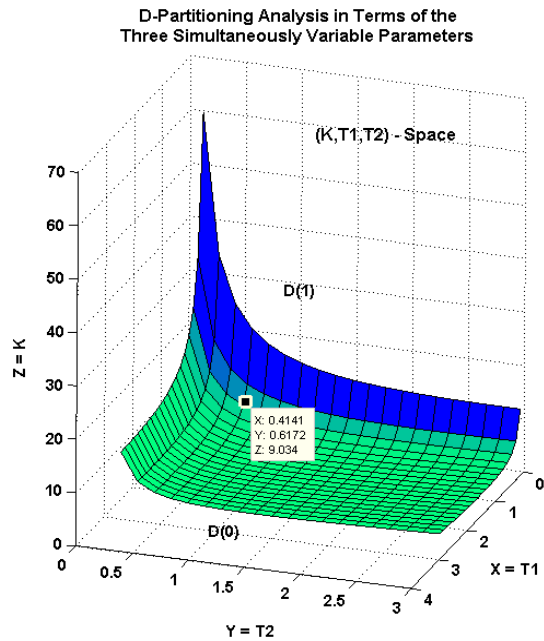


Figure 14. D-Partitioning in Case of a System with Three Simultaneously Variable Parameters

The D-Partitioning plane, as seen in Figure 14, is defining the border between the space of stability D(0), below the plane, and the space of instability D(1), above the plane, for the case of simultaneous variation of three system's parameters.

According to the Advanced D-Partitioning rule, in this case the space of stability is the one that remains always on the left-hand side of the D-Partitioning plane within the range of $\omega = \pm \omega_s/2 = \pm 2\pi/2T_s$. Following this rule, the stable space is proved to be the space D(0).

Any point of the D-partitioning plane represents the simultaneous marginal values of the three variable parameters, which is a unique advancement and in the theory of control systems stability analysis.

8. System Evaluation with Nyquist and Bode Stability Criteria (Case of Three Variables)

For the chosen point on the D-partitioning plane, shown in Figure 14, the system marginal gain is $K = Z = 9.034$ and the marginal values of the two time-constants are $T_1 = X = 0.4141$ sec and $T_2 = Y = 0.6172$ sec accordingly. To test the system's performance, its transfer function is introduced initially in the continuous-time domain and then converted into its digital equivalent. Substituting the three marginal values of the variable parameters in equation (29) results in:

$$G_{031}(s) = \frac{9.034}{(1 + 0.4141s)(1 + 0.6172s)(1 + 0.8s)} \approx \frac{9.034}{0.192s^3 + 1.04s^2 + 1.8s + 1} \quad (33)$$

The system stability within the space D(0) is explored in the discrete-time domain if the following values are allocated to the variable parameters $K = 3$, $T_1 = 0.4141$ sec and $T_2 = 0.6172$ sec, resulting in equation (34).

$$G_{032}(s) = \frac{3}{(1 + 0.4141s)(1 + 0.6172s)(1 + 0.8s)} \approx \frac{3}{0.192s^3 + 1.04s^2 + 1.8s + 1} \quad (34)$$

The system instability within the space D(1) is explored in the discrete-time domain if $K = 20$, $T_1 = 0.4141$ sec and $T_2 = 0.6172$ sec, resulting in equation (35).

$$G_{033}(s) = \frac{20}{(1 + 0.4141s)(1 + 0.6172s)(1 + 0.8s)} \approx \frac{20}{0.192s^3 + 1.04s^2 + 1.8s + 1} \quad (35)$$

Then the system's Nyquist and Bode stability assessment in the discrete-time domain is achieved by the code:

```
>> Go31=tf([0 9.034],[0.192 1.04 1.8 1])
Transfer function:
          9.034
-----
0.192 s^3 + 1.04 s^2 + 1.8 s + 1
>> God31=c2d(Go31,0.05,'tustin')
Transfer function:
0.0006417 z^3 + 0.001925 z^2 + 0.001925 z + 0.0006417
-----
z^3 - 2.742 z^2 + 2.505 z - 0.7626
>> Go32=tf([0 3],[0.192 1.04 1.8 1])
Transfer function:
          3
-----
0.192 s^3 + 1.04 s^2 + 1.8 s + 1
>> God32=c2d(Go32,0.05,'tustin')
Transfer function:
0.0002139 z^3 + 0.0006417 z^2 + 0.0006417 z + 0.0002139
-----
z^3 - 2.742 z^2 + 2.505 z - 0.7626
>> Go33=tf([0 20],[0.192 1.04 1.8 1])
Transfer function:
          20
-----
0.192 s^3 + 1.04 s^2 + 1.8 s + 1
>> God33=c2d(Go33,0.05,'tustin')
Transfer function:
0.001426 z^3 + 0.004278 z^2 + 0.004278 z + 0.001426
-----
z^3 - 2.742 z^2 + 2.505 z - 0.7626
Sampling time: 0.05
>> nyquist(God31,God32,God33)
>> margin(God31)
GM = -0.224 dB, PM = -0.921 deg
>> margin(God32)
GM = 9.3 dB, PM = 45 deg
>> margin(God33)
GM = -7.18 dB, PM = -22.8 deg
```

Nyquist Diagrams Proving the D-Partitioning for Cases of Marginal, Stable and Unstable System with Three Variables

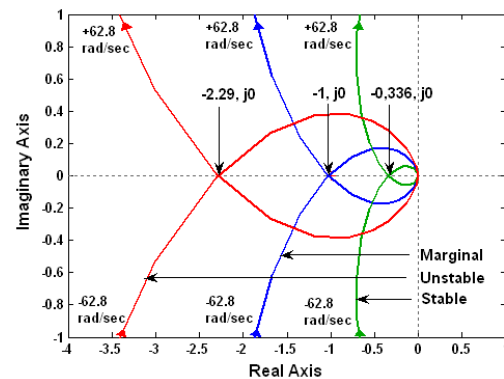


Figure 15. Prove of the D-Partitioning Analysis – Cases of Marginal, Stable and Unstable System with Three Variable Parameters



Applying the Nyquist stability criterion for the cases of marginal, stable and unstable system with three variable parameters, prove exactly the results from the Advanced D-Partitioning in the discrete time-domain.

The system's marginal case is also confirmed with the aid of the Bode stability criterion as shown in Figure 16.

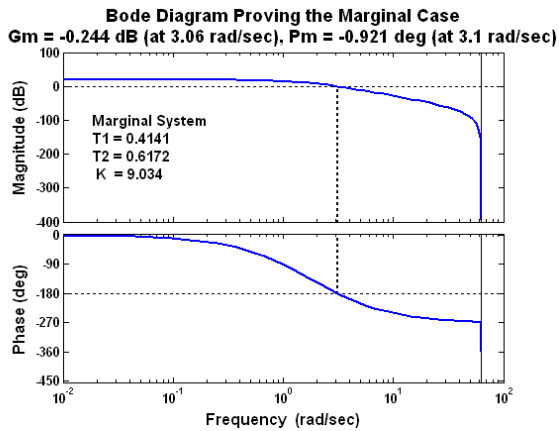


Figure 16. Prove of the D-Partitioning Analysis – Case Marginal System (Case $T_1 = 0.4141$ sec, $T_2 = 0.6172$ sec and $K = 9.034$)

If the variable parameters are $K = 9.034$, $T_1 = 0.4141$ sec and $T_2 = 0.6172$ sec, the results for the digital system gain margin is $GM = -0.244$ dB ≈ 0 dB and the system phase margin is $PM = -0.921^\circ \approx 0^\circ$, as seen in the Bode diagrams in Figure 16. This proves that the three variable parameters have simultaneously marginal values.

The system stability within the space $D(0)$ is explored in the discrete-time domain if the following values are allocated to the variable parameters $K = 3$, $T_1 = 0.4141$ sec and $T_2 = 0.6172$ sec. As seen in the code, the positive values of the gain and phase margins ($GM = 9.3$ dB and $PM = 45^\circ$) prove that if the system's gain is reduced to $K = 3$, while keeping the values of the time constants, the digital control system will operate in the space of stability $D(0)$. Similar result can be achieved if any of the two time constant values is reduced.

The system instability within the space $D(1)$ is explored in the discrete-time domain if $K = 20$, $T_1 = 0.4141$ sec and $T_2 = 0.6172$ sec. From the code above, the negative values of the gain and phase margins ($GM = -7.18$ dB and $PM = -22.8^\circ$) prove that if the system's gain is increased to $K = 20$, while keeping the values of the time constants, the digital control system operates in the space of instability $D(1)$. Similar result can be achieved if any of the two time constant values is increased.

9. Conclusion

The main contribution of this research is the achieved further development of the Advanced D-Partitioning method in its application for the analysis of digital control systems with multivariable parameters. Cases of systems with one, two or three simultaneously variable parameters are illustrated. There is a close match between the discrete and the continuous-time system performance if the Euler's approximation is observed.

The successful application of the analysis method in the discrete-time domain, by employing the Bilinear Tustin

Transform, demonstrates that the basic concept of the Advanced D-Partitioning is relevant and valid for digital control systems as well. The control system's discretization, based on the Bilinear Tustin Transform is providing the most precise results.

When the Advanced D-Partitioning is compared with the Nyquist and the Bode stability criteria, even if both these methods are applicable for continuous and for digital control systems, their disadvantage is that they cannot directly establish the marginal value of a variable parameter. In this case, they can be used only to confirm the results from the Advanced D-Partitioning method.

The research further contributes to knowledge, since the Advanced D-Partitioning is the only method of stability analysis that in terms of its results illustrates graphically the regions of stability and instability in case of system's multivariable parameters. It was demonstrated that any point of the D-partitioning curve or D-partitioning plane, represents the simultaneous marginal values of the two or three variable parameters. This is a unique innovation in the theory of control systems stability analysis.

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Biographies



Prof. Kamen Yanev has a M.Sc. in Control Systems Engineering. His Ph.D. is in the area of Systems with Variable Parameters and Robust Control Design. He started his academic career in 1974, following a number of academic promotions and being involved in considerable research, service and teaching at different Commonwealth Universities around the world. Prof. Yanev worked in a number of outstanding universities in Europe and in Africa, in countries like Nigeria, Zimbabwe and Botswana. Currently he works as Associate Professor in Control and Instrumentation Engineering at the Department of Electrical Engineering at University of Botswana.

His major research is in the field of Control Systems Engineering as well as in the subjects of Electronics and Instrumentation. He has more than 118 publications in international journals and conference proceedings. Most of his latest publications and current research interests are in the field of Electronics, Analysis of Control Systems with Variable Parameters and Robust Control Design.

Prof. Yanev is interacting with Saint-Cyr French Military Academy, supervising Military Academy students.

He is a member of the Academic Community of International Congress for Global Science and Technology (ICGST), a member of the Instrumentation, Systems and Automation Society (ISA), a member of the Institution of Electrical and Electronic Engineering (IEEE) and the Botswana Institute of Engineers (BIE).

Prof. Yanev is also a member of the Editorial Board of the International Journal of Energy Systems, Computers and Control (IJESCC), a member of the Editorial Board of the Journal of Multidisciplinary Engineering Science and Technology (JMEST), Germany, a reviewer for ACTA Press, Canada.

